Rossmoyne Senior High School

WA Exams Practice Paper C, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ____

Your name ____

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	54	35
Section Two: Calculator-assumed	13	13	100	96	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **65% (96 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(6 marks)**

A small ball is hit from a point on level ground with an initial velocity vector of $20\mathbf{i}+16\mathbf{j}$ metres per second, where $\mathbf{i}$ and $\mathbf{j}$ are the horizontal and vertical components respectively. The ball lands at the same height as the point from which it was hit and an acceleration of $-10\mathbf{j}$ acts throughout its flight.

Determine

(a) the position vector of the ball after t seconds, relative to its initial position. (3 marks)

$\mathbf{a}=-10\mathbf{j}$

$\mathbf{v}=20\mathbf{i}+(16-10t)\mathbf{j}$ (given initial velocity of $20\mathbf{i}+16\mathbf{j}$)

$\mathbf{r}=(20t)\mathbf{i}+(16t-5t^2)\mathbf{j}$ (given initial position of $0\mathbf{i}+0\mathbf{j}$)

(b) the speed of the ball at the instant it reaches its maximum height. (1 mark)

20 m/s.

At max ht, vertical velocity is zero, so speed is just horizontal component.

(c) the distance of the ball from its initial position when it lands. (2 marks)

$16t-5t^2=0$

$t(16-5t)=0 \Rightarrow t=\cancel{0},\ 3.2$

$20\times 3.2=64$ m

Question 10 **(7 marks)**

(a) Find the exact volume of the solid of revolution formed when the line $3x+4y=36$ between the limits $y=1$ and $y=7$ is rotated about the y-axis. (3 marks)

$3x+4y=36 \Rightarrow x=12-\frac{4}{3}y$

$$V=\pi\int_1^7\left(12-\tfrac{4}{3}y\right)^2\ dy$$

$$=\frac{896\pi}{3} \text{ cubic units}$$

(b) When the same line, $3x+4y=36$, is rotated about the x-axis between the limits $x=1$ and $x=a$, the volume of the solid of revolution formed is $\frac{489\pi}{2}$. Determine the value of a.

(4 marks)

$3x+4y=36 \Rightarrow y=9-\frac{3}{4}x$

$$\frac{489\pi}{2}=\pi\int_1^a\left(9-\tfrac{3}{4}x\right)^2\ dx$$

$$\frac{489}{2}=\frac{4\left(\frac{3}{4}a-9\right)^3}{9}+\frac{3993}{16}$$

$a=9$

Question 11 **(6 marks)**

Let two complex numbers be $z = a(\cos\theta + i\sin\theta)$ and $w = b(\cos\phi + i\sin\phi)$.

Determine the following in terms of a, b, ϕ and θ.

(a) $Arg\left(\sqrt{w}\right)$. (1 mark)

$$\boxed{\frac{\phi}{2}}$$

(b) $\dfrac{|w|}{|z^2|}$. (1 mark)

$$\boxed{\frac{|w|}{|z^2|} = \frac{b}{a^2}}$$

(c) $Arg\left(\dfrac{zw}{i}\right)$. (2 marks)

$$i = cis\frac{\pi}{2}$$

$$Arg\left(\frac{zw}{i}\right) = \theta + \phi - \frac{\pi}{2}$$

(d) $\left|z^{-1}w^2\right|$. (2 marks)

$$\begin{aligned} \left|z^{-1}w^2\right| &= \left|z^{-1}\right| \times \left|w^2\right| \\ &= \frac{1}{a} \times b^2 \\ &= \frac{b^2}{a} \end{aligned}$$

Question 12 **(8 marks)**

A random sample of 45 households in a suburb was selected as part of a study on electricity consumption, and the number of kilowatt hours (kWh) was recorded for each household. The mean consumption was found to be 415 kWh. In a very large study in the previous year, it was found that the standard deviation of the consumption was 64 kWh.

(a) Calculate a 98% confidence interval for the mean electricity consumption of households in this suburb. (2 marks)

$$415 \pm 2.326\frac{64}{\sqrt{45}} = (392.8, 437.2)$$

(b) An electricity supplier claimed that the mean consumption of households in the suburb was 435 kWh. Does the sample above provide a strong reason to doubt this claim? Justify your answer. (2 marks)

No.

435 lies within the 98% confidence interval and so there is no reason to doubt claim.

(c) 36 similar studies are planned for the suburb.

(i) Determine the number of households that should be sampled in each of these studies to be 95% confident that the mean electricity consumption of households is within 15 kWh of the true value. (2 marks)

$$n = \left(\frac{1.96 \times 64}{15}\right)^2$$
$$= 69.9$$

Sample 70 households.

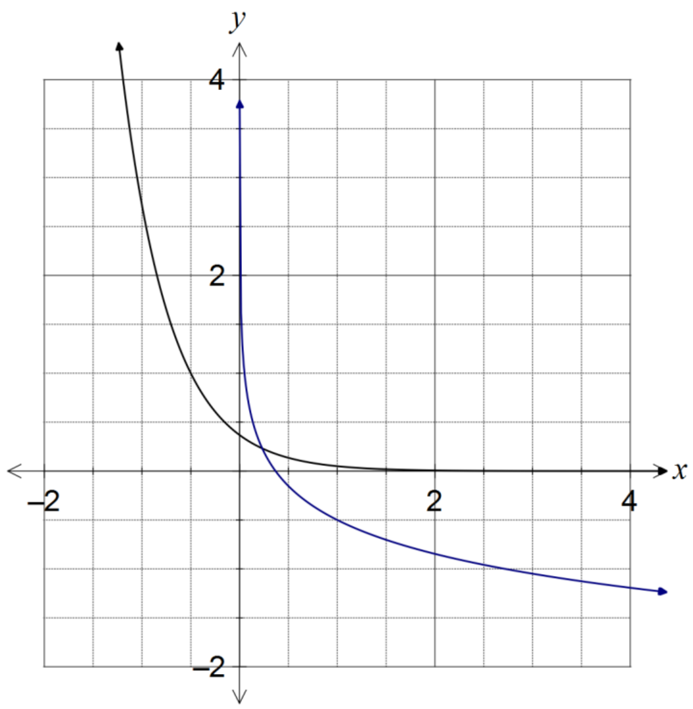
(ii) How many of the 95% confidence intervals from these additional studies are not expected to contain the true mean? Justify your answer. (2 marks)

$$0.05 \times 36 = 1.8$$

Two of the 36 studies, as we expect 95% of the intervals to contain the true mean.

Question 13 **(7 marks)**

Two functions are given by $f(x) = e^{-x}$ and $g(x) = 2x + 1$. The graph of their composite, $y = f \circ g(x)$, is shown below.



(a) Determine $f \circ g(x)$ and state the range of the composite function. (2 marks)

$f \circ g(x) = e^{-(2x+1)}$

Range: $y > 0$

(b) Determine $g^{-1} \circ f^{-1}(x)$, noting any domain restrictions. (3 marks)

$f^{-1}(x) = -\ln(x)$

$g^{-1}(x) = \frac{1}{2}(y-1)$

$g^{-1} \circ f^{-1}(x) = \frac{1}{2}(-\ln(x) - 1) = \ln\left(\frac{1}{\sqrt{x}}\right) - \frac{1}{2}$

Domain: $x > 0$

(c) Sketch the graph of $y = g^{-1} \circ f^{-1}(x)$ on the axes above. (2 marks)

Question 14 **(11 marks)**

(a) Determine the radius and coordinates of the centre of the sphere with equation $x^2 + y^2 + z^2 - 2x + 4y - 4z = 40$. (2 marks)

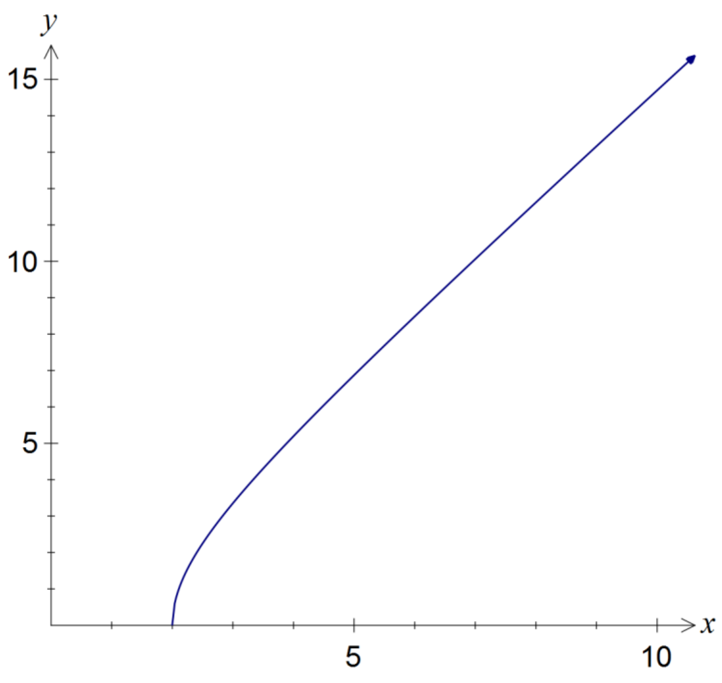
$$x^2 + y^2 + z^2 - 2x + 4y - 4z = 40$$
$$(x-1)^2 + (y+2)^2 + (z-2)^2 = 40 + 1 + 4 + 4$$
$$= 7^2$$

Centre is (1, -2, 2) and radius is 7.

(b) Determine the Cartesian equation of the hyperbola with vector equation given by $\mathbf{r} = (2\sec t)\mathbf{i} + (3\tan t)\mathbf{j}$, and sketch the part of the path in the first quadrant on the axes below. (4 marks)

$$\frac{x}{2} = \sec t,\ \frac{y}{3} = \tan t \Rightarrow 1 + \left(\frac{y}{3}\right)^2 = \left(\frac{x}{2}\right)^2$$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$



(c) Determine the vector equation of the straight line that passes through the two points with coordinates (-5, 3, 7) and (1, -2, 10). (2 marks)

$$\begin{bmatrix} 1 \\ -2 \\ 10 \end{bmatrix} - \begin{bmatrix} -5 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 3 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 1 \\ -2 \\ 10 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ -5 \\ 3 \end{bmatrix}$$

(d) Determine a vector equation, in the form $\mathbf{r} \bullet \mathbf{n} = c$, of the plane that contains the straight line from (c) and also the point (3, 3, 3). (3 marks)

$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ -5 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 20 \\ 48 \\ 40 \end{bmatrix} = 4\begin{bmatrix} 5 \\ 12 \\ 10 \end{bmatrix}$$

$$\mathbf{r} \bullet \begin{bmatrix} 5 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ 12 \\ 10 \end{bmatrix} = 81 \Rightarrow \mathbf{r} \bullet \begin{bmatrix} 5 \\ 12 \\ 10 \end{bmatrix} = 81$$

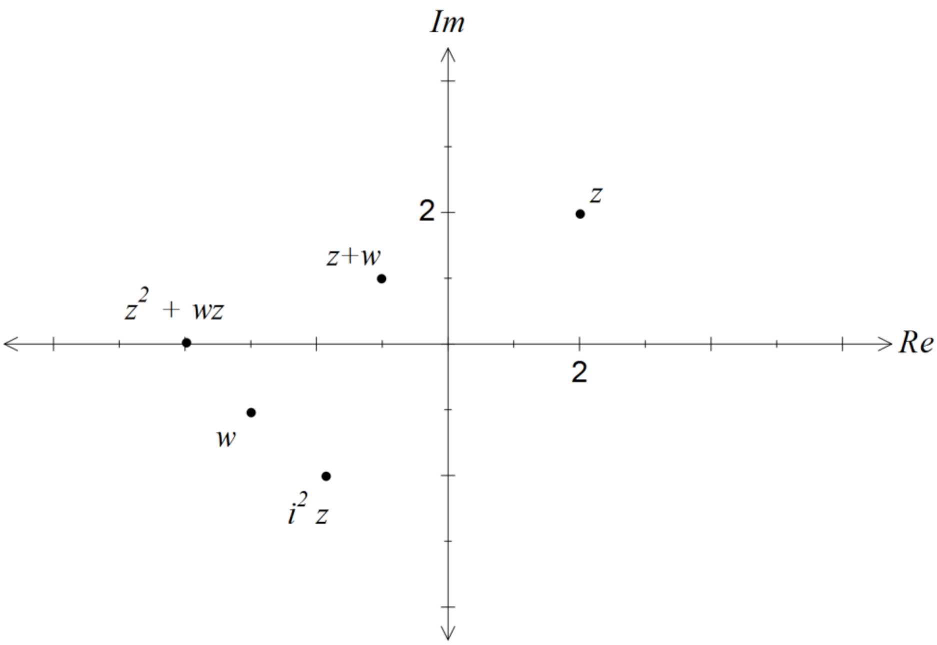
Question 15 **(8 marks)**

(a) The complex numbers z and $z+w$ are shown on the Argand diagram below. On the same diagram plot and label the location of

(i) i^2z. (1 mark)

(ii) w. (1 mark)

(iii) z^2+wz. (2 marks)



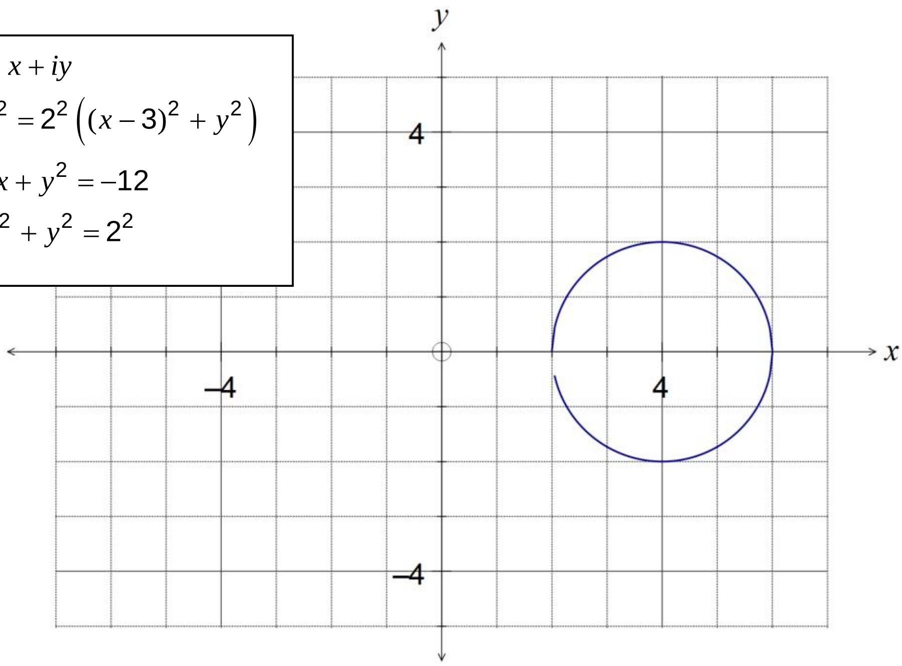
(b) Sketch the region $|z| = 2|z-3|$ in the complex plane. (4 marks)

Let $z = x+iy$

$x^2+y^2 = 2^2\left((x-3)^2+y^2\right)$

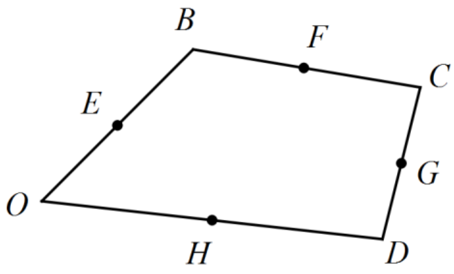
$x^2-8x+y^2=-12$

$(x-4)^2+y^2=2^2$



Question 16 **(7 marks)**

In the diagram below, E, F, G and H are midpoints of the sides of the quadrilateral $OBCD$.



Let $\overrightarrow{OB} = 2\mathbf{b}$, $\overrightarrow{OC} = 2\mathbf{c}$ and $\overrightarrow{OD} = 2\mathbf{d}$.

(a) Show that $\overrightarrow{OF} = \mathbf{b} + \mathbf{c}$. (2 marks)

$$\begin{aligned}\overrightarrow{OF} &= \overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{BC} \\ &= \overrightarrow{OB} + \tfrac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OB}\right) \\ &= 2\mathbf{b} + \tfrac{1}{2}(2\mathbf{c} - 2\mathbf{b}) \\ &= \mathbf{b} + \mathbf{c}\end{aligned}$$

(b) Determine $\overrightarrow{OG}$ in terms of $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. (2 marks)

$$\begin{aligned}\overrightarrow{OG} &= \overrightarrow{OD} + \tfrac{1}{2}\overrightarrow{DC} \\ &= \overrightarrow{OD} + \tfrac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OD}\right) \\ &= 2\mathbf{d} + \tfrac{1}{2}(2\mathbf{c} - 2\mathbf{d}) \\ &= \mathbf{d} + \mathbf{c}\end{aligned}$$

(c) Prove that $EFGH$ is a parallelogram. (3 marks)

$$\begin{aligned}\overrightarrow{EF} &= \overrightarrow{OF} - \overrightarrow{OE} \\ &= \mathbf{b} + \mathbf{c} - \mathbf{b} \\ &= \mathbf{c}\end{aligned}$$

$$\begin{aligned}\overrightarrow{HG} &= \overrightarrow{OG} - \overrightarrow{OH} \\ &= \mathbf{d} + \mathbf{c} - \mathbf{d} \\ &= \mathbf{c}\end{aligned}$$

Hence EF is parallel to HG and of equal length and so $EFGH$ must be a parallelogram.

Question 17 **(8 marks)**

(a) Solve the equation $z^4 = 2 + 2\sqrt{3}i$, expressing all solutions in polar form. (4 marks)

$$z^4 = 2^2 cis(\tfrac{\pi}{3})$$

$$z_1 = \left(2^2\right)^{\frac{1}{4}} cis\left(\frac{\pi}{3} \times \frac{1}{4}\right)$$

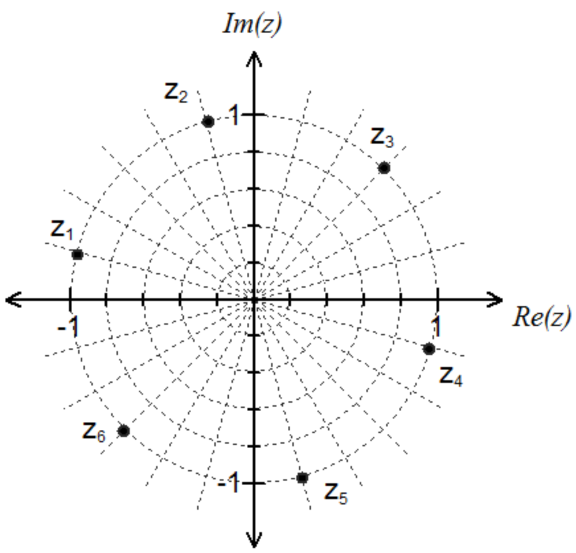
$$z_1 = \sqrt{2}cis(\tfrac{\pi}{12})$$

$$z_2 = \sqrt{2}cis(\tfrac{7\pi}{12})$$

$$z_3 = \sqrt{2}cis(-\tfrac{5\pi}{12})$$

$$z_4 = \sqrt{2}cis(-\tfrac{11\pi}{12})$$

(b) One solution to the equation $z^6 = a + bi$, where a and b are real constants, is shown on the diagram below.



(i) Plot all other solutions to the equation on the diagram. (2 marks)

(ii) Determine the values of a and b. (2 marks)

Using $z_3 = cis(\tfrac{\pi}{4})$

$$z^6 = cis(\tfrac{6\pi}{4})$$

$$= -i$$

$$a = 0,\ b = -1$$

Question 18 **(6 marks)**

(a) A small body moves along the curve with equation $y = \dfrac{1}{1+0.01x^2}$, where x and y are measured in centimetres. Determine the rate at which the body's y-coordinate is changing when the it passes through the point (-10, 0.5), given that its x-coordinate is increasing at 2 metres per second. (3 marks)

$$\frac{dx}{dt} = 200$$

$$\frac{dy}{dx} = \left.\frac{-(0.02x)}{\left(1+0.01x^2\right)^2}\right|_{x=-10}$$

$$= 0.05$$

$$\frac{dy}{dt} = 0.05 \times 200$$

$$= 10 \text{ cm/s}$$

(b) The differential equation for a curve passing through the point (1, -1) is given by $\dfrac{dy}{dx} = xy - x^2$. Use the incremental formula $\delta y = \dfrac{dy}{dx} \times \delta x$, with $\delta x = 0.2$, to calculate an estimate for the y-coordinate of the curve when $x = 1.4$. (3 marks)

Using $\delta x = 0.2$ and Euler's method:

x	y	$\frac{dy}{dx}$	$\delta y = \frac{dy}{dx} \times \delta x$
1	-1	-2	-0.4
1.2	-1.4	-3.12	-0.624
1.4	-7.64		

Estimate is $y = -7.64$.

Question 19 **(8 marks)**

(a) Show that if $P = \dfrac{a}{b + ke^{-at}}$, where a, b and k are positive constants, then $\dfrac{dP}{dt} = aP - bP^2$.

(4 marks)

Use these results:

$ke^{-at} = \dfrac{a}{P} - b$ and $P = a\left(b + ke^{-at}\right)^{-1}$

$$\begin{aligned}\frac{dP}{dt} &= a\left(-1\times -ake^{-at}\right)\left(b+ke^{-at}\right)^{-2}\\ &= \frac{a^2ke^{-at}}{\left(b+ke^{-at}\right)^2}\\ &= \frac{a^2\left(\dfrac{a}{P}-b\right)}{\left(a/P\right)^2}\\ &= aP - bP^2\end{aligned}$$

(b) Following the accidental release of eight vermin on an island, the ensuing growth of their population was modelled by $\dfrac{dP}{dt} = 0.2P - 0.0005P^2$, where P is the number of vermin t months after their release. Use your result from (a) to express P as a function of t.

(2 marks)

$t = 0,\ \dfrac{0.2}{0.0005 + ke^0} = 8 \Rightarrow k = 0.0245$

$$P = \frac{0.2}{0.0005 + 0.0245e^{-0.2t}}$$

(c) Calculate the number of months taken for the size of the vermin population to reach 75% of its long term size. (2 marks)

$$t \to \infty,\ P \to \frac{0.2}{0.0005} = 400$$

$$\frac{0.2}{0.0005 + 0.0245e^{-0.2t}} = 400 \times 0.75 \ \Rightarrow\ t = 24.95$$

Take 25 months.

Question 20 **(7 marks)**

On the basis of the results obtained from a random sample of 81 bags produced by a mill, the 95% confidence interval for the mean weight of flour in a bag is found to be (514.56 g, 520.44 g).

(a) Find the value of $\bar{x}$, the mean weight of the sample. (1 mark)

$$\bar{x} = \frac{514.56 + 520.44}{2} = 517.5 \text{ g}$$

(b) Find the value of σ, the standard deviation of the normal population from which the sample is drawn. (2 marks)

$$520.44 - 517.5 = 1.96\frac{\sigma}{\sqrt{81}}$$

$$\sigma = 13.5 \text{ g}$$

(c) Calculate the 99% confidence interval for the mean weight of flour in a bag. (2 marks)

$$517.5 \pm 2.576\frac{13.5}{\sqrt{81}}$$

$$= (513.636 \text{ g},\ 521.364 \text{ g})$$

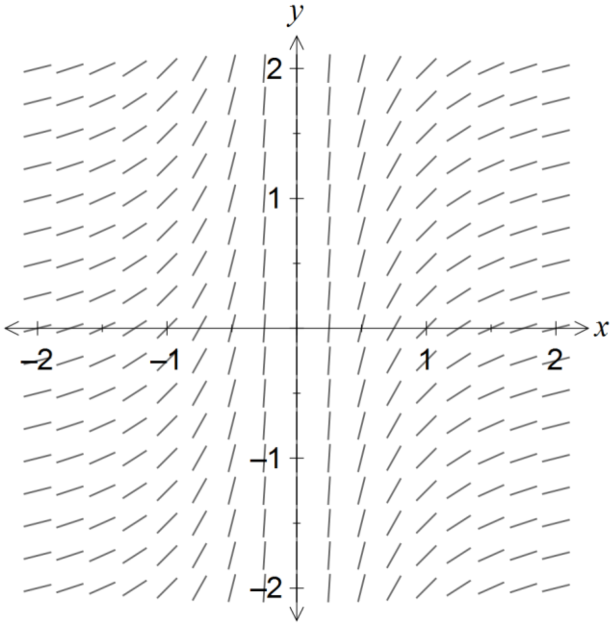
(d) Using the sample mean from (a) as the best estimate for the population mean, what is the probability that the sample mean of a larger sample of 225 bags is less than 516 g? (2 marks)

$$X \sim N\left(517.5, \frac{13.5^2}{225}\right)$$

$$P(X < 516) = 0.0478$$

Question 21 **(7 marks)**

The gradient field of a first-order differential equation is shown below.



(a) Use the scale shown to determine a general differential equation that would result in this slope field. (2 marks)

$$\frac{dy}{dx} = \frac{a}{x^2} + b$$

(b) State **two** reasons for your answer in (a). (2 marks)

- gradient is independent of y.
- power of x must be even.
- y-axis is vertical asymptote.
- etc

(c) Determine a possible general equation for y, noting where restrictions exist on the values of any constants used. (2 marks)

$$y = \frac{c_1}{x} + c_2 x + c_3$$

$$c_1 < 0$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________